SYMMETRIC STABLE PROCESSES IN PARABOLA–SHAPED REGIONS

RODRIGO BAÑUELOS AND KRZYSZTOF BOGDAN

Abstract. We identify the critical exponent of integrability of the first exit time of rotation invariant stable Lévy process from parabola–shaped region.

1. Introduction

For $d = 2, 3, \dots$ and $0 < \beta < 1$, we define the *parabola–shaped region* in $\mathbb{R}^d$

$$\mathcal{P}_\beta = \{x = (x_1, \tilde{x}) \, : \, x_1 > 0 \, , \, \tilde{x} \in \mathbb{R}^{d-1} \, , |\tilde{x}| < x_1^\beta\}.$$

Let $0 < \alpha < 2$. By $\{X_t\}$ we denote the isotropic α-stable $\mathbb{R}^d$-valued Lévy process ([17]). The process is time-homogeneous, has right-continuous trajectories with left limits, α-stable rotation invariant independent increments, and characteristic function

$$\mathbf{E}_x e^{i\xi(X_t - x)} = e^{-t|\xi|^\alpha}, \quad x \in \mathbb{R}^d \, , \quad \xi \in \mathbb{R}^d, \quad t \geq 0. \tag{1}$$

Here $\mathbf{E}_x$ is the expectation with respect to the distribution $\mathbf{P}_x$ of the process starting from $x \in \mathbb{R}^d$. For an open set $U \subset \mathbb{R}^d$, we define $\tau_U = \inf\{t \geq 0; \; X_t \notin U\}$, the first exit time of U ([17]). In the case of the parabola-shaped region $\mathcal{P}_\beta$, we simply write τ_β for $\tau_{\mathcal{P}_\beta}$.

The main result of this note is the following result.

Theorem 1. *Let $p \geq 0$. Then $E_x {\tau_\beta}^p < \infty$ for (some, hence for all) $x \in \mathcal{P}_\beta$ if and only if $p < p_0$, where*

$$p_0 = \frac{(d-1)(1-\beta) + \alpha}{\alpha\beta}.$$

Theorem 1 may be regarded as an addition to the research direction initiated in [3], where it was proved that for $\beta = 1/2$, $d = 2$ and $\{X_t\}$ replaced by the *Brownian motion process* $\{B_t\}$, τ_β is subexponentially integrable. For

The first author was supported in part by NSF grant # 9700585-DMS
The second author was supported in part by KBN (2P03A 041 22) and by RTN (HPRN-CT-2001-00273-HARP)
2000 Mathematics Subject Classification: Primary 31B05, 60J45.
Key words and phrases: symmetric stable process, parabola, exit time, harmonic measure.
Date: 7/14/2004.

$\{B_t\}$, the generalizations to all the considered domains $\mathcal{P}_\beta$ (along with essential strengthening of the result of [3]) were subsequently obtained in [14], [15], [5], [11] and [2]. We should note that this direction of research was influenced to a large degree by the result of Burkholder [10] on the critical order of integrability of the exit times of $\{B_t\}$ from cones. For more on the many generalizations of Burkholder's result we refer the reader to [4] and references therein.

Brownian motion is a limiting case of the isotropic α-stable process and B_{2t} corresponds to $\alpha = 2$ via the analogue of (1). Extension of some of the above-mentioned results pertaining to cones to the case $0 < \alpha < 2$ (that is, for $\{X_t\}$) were given in [12], [13], [16] and [1], see also [7]. It should be emphasized that while there are many similarities between $\{X_t\}$ and $\{B_{2t}\}$, there also exist essential differences in their respective properties and their proofs. For example the critical exponent of integrability of the exit time of $\{X_t\}$ is less than 1 for every cone, however narrow it may be ([1]), while it is arbitrarily large for $\{B_t\}$ in sufficiently narrow cones ([10]). A similar remark applies to regions $\mathcal{P}_\beta$. The critical exponent of integrability of τ_β for $\{X_t\}$ given in Theorem 1 is *qualitatively different* from that of $\{B_t\}$ ([2]).

Finally, we have recently learned that Pedro J. Méndez-Hernández, in a paper in preparation, has obtained results similar to those presented. His results apply to more general regions defined by other "slowly" increasing functions.

The proof of Theorem 1 is based on estimates for harmonic measure of $\{X_t\}$, following the general idea used in [2] for $\{B_t\}$. The necessity of the condition $p < p_0$ for finiteness of $E_x \tau_\beta{}^p$ is proved in Lemma 2 and the sufficiency is proved in Lemma 6. The main technical result of the paper is Lemma 5 where we prove sharp estimates for harmonic measure of cut-offs of the parabola-shaped region. At the end of the paper we discuss some remaining open problems.

2. Proofs

We begin by reviewing the notation. By $|\cdot|$ we denote the Euclidean norm in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, $r > 0$, and a set $A \subset \mathbb{R}^d$ we let $B(x,r) = \{y \in \mathbb{R}^d \colon |x-y| < r\}$ and $\operatorname{dist}(A,x) = \inf\{|x-y| \colon y \in A\}$. A^c is the complement of A. We always assume Borel measurability of the considered sets and functions. In equalities and inequalities, unless stated otherwise, c denotes some unspecified positive real number whose value depends *only* on d, α and β.

It is well known that $(X_t, \mathbf{P}_x)$ is a strong Markov processes with respect to the standard filtration ([17]).

For an open set $U \subset \mathbb{R}^d$, with exit time τ_U, the $\mathbf{P}_x$ distribution of X_{τ_U}:

$$\omega_U^x(A) := P_x\{X_{\tau_U} \in A, \tau_U < +\infty\}\,, \quad A \subset \mathbb{R}^d\,,$$

is a subprobability measure concentrated on U^c (probability measure if U is bounded) called the *harmonic measure*. When $r > 0$, $|x| < r$ and $B = B(0,r) \subset \mathbb{R}^d$, the corresponding harmonic measure has the density function $P_r(x,y) = d\omega_B^x/dy$ (the *Poisson kernel* for the ball). We have

$$P_r(x,y) = C(d,\alpha)\left[\frac{r^2-|x|^2}{|y|^2-r^2}\right]^{\alpha/2} |y-x|^{-d} \quad \text{if} \quad |y| > r, \tag{2}$$

where $C(d,\alpha) = \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$, and $P_r(x,y) = 0$ otherwise ([6]).

The *scaling* property of X_t plays a role in this paper. Namely, for every $r > 0$ and $x \in \mathbb{R}^d$ the $\mathbf{P}_x$ distribution of $\{X_t\}$ is the same as the $\mathbf{P}_{rx}$ distribution of $\{r^{-1}X_{r^\alpha t}\}$ (see (1)). In particular, the $\mathbf{P}_x$ distribution of τ_U is the same as the $\mathbf{P}_{rx}$ distribution of $r^{-\alpha}\tau_{rU}$. In short, $\tau_{rU} = r^\alpha \tau_U$ in distribution.

Let ${\mathcal{P}_\beta}' = \mathcal{P}_\beta \cap \{x_1 > 1\,, |\tilde{x}| < x_1^\beta/2\}$. We claim that if $x \in {\mathcal{P}_\beta}'$ then $B(x,|x|^\beta/5) \subset \mathcal{P}_\beta$. Indeed, let $x = (x_1,\tilde{x}) \in {\mathcal{P}_\beta}'$ and $y = (y_1,\tilde{y}) \in B(x,|x|^\beta/5)$. We have $|\tilde{x}| < x_1^\beta/2 < x_1/2$ hence

$$1 < x_1 \le |x| = \sqrt{x_1^2 + |\tilde{x}|^2} < \sqrt{5/4}x_1 < 5x_1/4\,.$$

Then

$$y_1 \ge x_1 - |x|^\beta/5 > x_1 - |x|/5 > x_1 - (5/4)x_1/5 = 3x_1/4\,,$$

and so

$$|\tilde{y}| \le |\tilde{x}| + |x|^\beta/5 < x_1^\beta/2 + (5x_1/4)^\beta/5 < 3x_1^\beta/4 < 3(4y_1/3)^\beta/4 < y_1^\beta\,,$$

which yields $y \in \mathcal{P}_\beta$, as claimed.

Lemma 2. *If $p \ge p_0$ then $E_x{\tau_\beta}^p = \infty$ for every $x \in \mathcal{P}_\beta$.*

Proof. Define $\tau = \inf\{t \ge 0 : X_t \notin B(X_0,|X_0|^\beta/5)\}$. For $y \in \mathbb{R}^d$ and (nonnegative) p by scaling we have $\mathbf{E}_y\tau^p = \mathbf{E}_0\tau^p_{B(0,1)}(|y|^\beta/5)^{p\alpha} = c|y|^{\alpha\beta p}$. Let $x \in \mathcal{P}_\beta$, $r = \operatorname{dist}(x,{\mathcal{P}_\beta}^c)$, $B = B(x,r)$, and let R be so large that $B \subset \mathcal{P}_\beta \cap \{y_1 \le R\}$. By strong Markov property we have

$$\begin{aligned}
\mathbf{E}_x{\tau_\beta}^p &\ge \mathbf{E}_x\{X_{\tau_B} \in {\mathcal{P}_\beta}'\,;\ \mathbf{E}_{X_{\tau_B}}{\tau_\beta}^p\} \ge \mathbf{E}_x\{X_{\tau_B} \in {\mathcal{P}_\beta}'\,;\ \mathbf{E}_{X_{\tau_B}}\tau^p\} \\
&= c\mathbf{E}_x\{X_{\tau_B} \in {\mathcal{P}_\beta}'\,;\ |X_{\tau_B}|^{\alpha\beta p}\} \ge c\int_{{\mathcal{P}_\beta}'\cap\{y_1>R\}} P_r(0,y-x)|y|^{\alpha\beta p}\,dy \\
&\ge cr^\alpha C(d,\alpha)\int_R^\infty dy_1 \int_{|\tilde{y}|<y_1^\beta/2} |y-x|^{-d-\alpha}|y|^{\alpha\beta p}\,d\tilde{y} \\
&\ge \mathit{const.}\int_R^\infty y_1^{\beta(d-1)+\alpha\beta p-d-\alpha}\,dy_1\,.
\end{aligned}$$

If $p \ge p_0$, then $\beta(d-1)+\alpha\beta p-d-\alpha \ge \beta(d-1)+(d-1)(1-\beta)+\alpha-d-\alpha = -1$ and the last integral is positive infinity, proving the lemma. $\square$

For $0 \le u < v \le \infty$, we let ${\mathcal{P}_\beta}^{u,v} = \mathcal{P}_\beta \cap \{(x_1, \tilde{x}) \,:\, u \le x_1 < v\}$. Let $0 < s < \infty$ and define $\tau_s = \tau(s) = \tau_{{\mathcal{P}_\beta}^{0,s}}$. Consider the cylinder

$$\mathcal{C} = \{x = (x_1, \tilde{x}) \in \mathbb{R}^d \,:\, -\infty < x_1 < s\,,\ |\tilde{x}| < s^\beta\}\,. \tag{3}$$

Clearly for $A \subset \mathcal{P}_\beta$,

$$\mathbf{P}_x\{X_{\tau_s} \in A\} \le \mathbf{P}_x\{X_{\tau_\mathcal{C}} \in A\}\,, \quad x \in \mathbb{R}^d\,. \tag{4}$$

By Lemma 4.3 of [9] for $A \subset \mathbb{R}^d \cap \{z_1 \ge s\}$ we have

$$\mathbf{P}_x\{X_{\tau_\mathcal{C}} \in A\} \le c \int_{z=(z_1,\tilde{z})\in A} \frac{s^{\alpha\beta}}{|z-x|^{d+\alpha}} \left[\frac{s^{\alpha\beta/2}}{(z_1-s)^{\alpha/2}} \vee 1\right] dz\,, \quad x \in \mathcal{C}. \tag{5}$$

The following lemma will simplify the use of the estimate (5).

Lemma 3. *Let $s \ge 1$ and $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$ with $x_1 \le s/2$. Let $s \le u < v \le \infty$ and assume that either $u \ge s + s^\beta$ or $u = s$ and $v \ge s + s^\beta$. Then*

$$\mathbf{P}_x\{X_{\tau_s} \in {\mathcal{P}_\beta}^{u,v}\} \le c s^{\alpha\beta} \int_u^v t^{-\alpha\beta p_0 - 1} dt\,. \tag{6}$$

Proof. Denote

$$f(y) = \frac{d\omega^x_{{\mathcal{P}_\beta}^{0,s}}}{dy}(y)\,.$$

Let $y = (y_1, \tilde{y}) \in {\mathcal{P}_\beta}^{s,\infty}$. From (4) and (5) we can conclude that

$$f(y) \le c\frac{s^{\alpha\beta}}{|y-x|^{d+\alpha}} \left[\frac{s^{\alpha\beta/2}}{(y_1-s)^{\alpha/2}} \vee 1\right]\,.$$

Since $s \ge 1$, we have $|\tilde{y}| \le y_1$ and $|y| \le \sqrt{2} y_1$. As $|y - x| \ge y_1 - x_1 \ge y_1/2$,

$$f(y) \le c s^{\alpha\beta} y_1^{-d-\alpha} \left[\frac{s^{\alpha\beta/2}}{(y_1-s)^{\alpha/2}} \vee 1\right]\,.$$

If $u \ge s + s^\beta$ then

$$\begin{aligned}\mathbf{P}_x\{X_{\tau_s} \in {\mathcal{P}_\beta}^{u,v}\} &\le c s^{\alpha\beta} \int_u^v y_1^{-d-\alpha} \int_{|\tilde{y}|<y_1^\beta} d\tilde{y}\, dy_1 \\ &= c s^{\alpha\beta} \int_u^v t^{-d-\alpha+\beta(d-1)} dt = c s^{\alpha\beta} \int_u^v t^{-\alpha\beta p_0 - 1} dt\,.\end{aligned}$$

If $u = s$ and $v \ge s + s^\beta$, we consider

$$I(s) = s^{\alpha\beta} \int_s^{s+s^\beta} t^{-\alpha\beta p_0 - 1} dt\,,$$

and

$$II(s) = s^{\alpha\beta} \int_s^{s+s^\beta} t^{-\alpha\beta p_0 - 1} \frac{s^{\alpha\beta/2}}{(t-s)^{\alpha/2}} dt\,.$$

It is enough to verify that $II(s) \leq cI(s)$. Note that $s + s^\beta \leq 2s$, hence

$$I(s) \geq s^{\alpha\beta}(2s)^{-\alpha\beta p_0 - 1} s^\beta = cs^{-\alpha\beta p_0 + \alpha\beta + \beta - 1}.$$

Since

$$II(s) \leq s^{\alpha\beta} s^{-\alpha\beta p_0 - 1} s^{\alpha\beta/2} \int_0^{s^\beta} z^{-\alpha/2} dz = cs^{-\alpha\beta p_0 + \alpha\beta + \beta - 1},$$

we get $II(s) \leq cI(s)$. □

The following result is an immediate consequence of Lemma 3.

Lemma 4. *If $s \geq 1$, $x = (x_1, \tilde{x}) \in \mathbb{R}^d$ and $x_1 \leq s/2$, then*

$$\mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} \leq cs^{-\alpha\beta(p_0 - 1)}. \tag{7}$$

Estimate (7) is sharp if, for example, $x = (s/2, \tilde{0})$, where $\tilde{0} = (0, \ldots, 0) \in \mathbb{R}^{d-1}$. Indeed, let $B = B(x, cs^\beta) \subset {\mathcal{P}_\beta}^{0,s}$, where $c > 0$ is small enough. By (2) we have

$$\begin{aligned} \mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} &\geq \mathbf{P}_x\{X_{\tau_B} \in {\mathcal{P}_\beta}^{s,\infty}\} \\ &\geq c(s^\beta)^\alpha \int_s^\infty dz_1 \int_{\{\tilde{z} \in \mathbb{R}^{d-1} : |\tilde{z}| < z_1^\beta\}} |(z_1, \tilde{z}) - x|^{-d-\alpha} d\tilde{z} \\ &\geq cs^{\alpha\beta} \int_s^\infty t^{\beta(d-1)-d-\alpha} dt = cs^{-\alpha\beta(p_0-1)}. \qquad (8) \end{aligned}$$

We can, however, improve the estimate when x is small relative to s.

Lemma 5. *If $s \geq 1$, $x = (x_1, \tilde{x}) \in \mathbb{R}^d$ and $x_1 \leq s/2$, then*

$$\mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} \leq C(x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}. \tag{9}$$

Proof. If $s \leq S$, where $0 < S < \infty$ is fixed, then (9) trivially holds with $C = S^{\alpha\beta p_0}$, hence from now on we assume $s \geq S$, where $S = 2^{k_0}$ and $k_0 \geq 2$ is such that

$$\frac{4^{\alpha\beta} c}{\alpha\beta(p_0 - 1)} \left(2^{k_0}\right)^{-\alpha\beta(p_0-1)} < \frac{1}{2}.$$

The reason for this choice of k_0 will be made clear later on. Here and in what follows c is the constant of Lemma 3. We will prove (9) by induction. If $s/4 \leq x_1 \leq s/2$ then by Lemma 4

$$\mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} \leq c_1 s^{\alpha\beta} s^{-\alpha\beta p_0} \leq 4^{\alpha\beta} c_1 x_1^{\alpha\beta} s^{-\alpha\beta p_0} = c_1(x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}.$$

Assume that n is a natural number and (9) holds for all $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$ such that $s/2^{n+1} \leq x_1 \leq s/2$.

Let $s/2^n \geq 1$ and $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$ be such that $s/2^{n+2} \leq x_1 < s/2^{n+1}$. Note that $1 \leq s/2^n \leq 4x_1$ here. We have

$$\begin{aligned}
\mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} &\leq \mathbf{P}_x\{X_{\tau(s/2^n)} \in {\mathcal{P}_\beta}^{s/2,\infty}\} \\
&+ \mathbf{E}_x\{X_{\tau(s/2^n)} \in {\mathcal{P}_\beta}^{s/2^n,s/2};\ \mathbf{P}_{X_{\tau(s/2^n)}}\{X_{\tau_s} \in \mathcal{P}_\beta\}\} \\
&= I + II\,.
\end{aligned}$$

By Lemma 3,

$$\begin{aligned}
I &\leq c(s/2^n)^{\alpha\beta} \int_{s/2}^{\infty} t^{-\alpha\beta p_0 - 1} dt = c 2^{\alpha\beta p_0} (s/2^n)^{\alpha\beta} s^{-\alpha\beta p_0} \\
&\leq c 2^{\alpha\beta p_0 + 2\alpha} x_1^{\alpha\beta} s^{-\alpha\beta p_0} \leq c_2 (x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}\,.
\end{aligned}$$

By Lemma 3 and induction

$$\begin{aligned}
II &\leq c(s/2^n)^{\alpha\beta} \int_{s/2^n}^{s/2} t^{-\alpha\beta p_0 - 1} C t^{\alpha\beta} s^{-\alpha\beta p_0} dt \\
&\leq \frac{c}{\alpha\beta(p_0 - 1)} (s/2^n)^{-\alpha\beta(p_0-1)} C (s/2^n)^{\alpha\beta} s^{-\alpha\beta p_0} \\
&\leq \frac{4^{\alpha\beta} c}{\alpha\beta(p_0 - 1)} (s/2^n)^{-\alpha\beta(p_0-1)} C (x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}\,.
\end{aligned}$$

Thus

$$R = \frac{I + II}{(x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta p_0}} \leq c_2 + \frac{4^{\alpha\beta} c}{\alpha\beta(p_0 - 1)} (s/2^n)^{-\alpha\beta(p_0-1)} C\,.$$

For n such that $s/2^n \geq 2^{k_0}$ we have $R \leq c_2 + C/2$ and we can take $C = 2(c_1 \vee c_2)$ in our inductive assumption to the effect that $R \leq c_2 + c_1 \vee c_2 \leq C$, and so (9) holds for every $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$, satisfying $s/2^{n+2} \leq x_1 \leq s/2$.

By induction, (9) is true with $C = 2(c_1 \vee c_2)$ for all $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$, satisfying $2^{k_0 - 2} \leq x_1 \leq s/2$.

If $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$ and $0 < x_1 \leq 2^{k_0-2}$, then

$$\begin{aligned}
\mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} &\leq \mathbf{P}_x\{X_{\tau(2^{k_0}/2)} \in {\mathcal{P}_\beta}^{s/2,\infty}\} \\
&+ \mathbf{E}_x\{X_{\tau(2^{k_0}/2)} \in {\mathcal{P}_\beta}^{2^{k_0}/2,s/2};\ \mathbf{P}_{X_{\tau(2^{k_0}/2)}}\{X_{\tau_s} \in \mathcal{P}_\beta\}\} \\
&\leq c(2^{k_0}/2)^{\alpha\beta} \int_{s/2}^{\infty} t^{-\alpha\beta p_0 - 1} dt \\
&+ c(2^{k_0}/2)^{\alpha\beta} \int_{2^{k_0}/2}^{s/2} t^{-\alpha\beta p_0 - 1} 2(c_1 \vee c_2) t^{\alpha\beta} s^{-\alpha\beta p_0} dt \\
&\leq c_3 s^{-\alpha\beta p_0} \leq c_3 (x_1 \vee 1)^{\alpha\beta} s^{-\alpha\beta(p_0-1)}\,.
\end{aligned}$$

We used here our previous estimates. The proof is complete. □

Note that (9) is sharp if, for example, $x = (x_1, \tilde{0})$ and $1 \le x_1 \le s/2$. This can be verified as in (8) to the effect that for such x the upper bound in Lemma 5 is of the same order as $\mathbf{P}_x\{X_{\tau_B} \in {\mathcal{P}_\beta}^{s,\infty}\}$, where B is the largest ball centered at x such that $B \subset \mathcal{P}_\beta$.

Informally, $\{X_t\}$ goes to ${\mathcal{P}_\beta}^{s,\infty}$ "mostly" by a direct jump from B.

This informal rule seems to be related to a "thinness" of $\mathcal{P}_\beta$ at infinity (or its inversion [8] at 0). This is false for cones ([1]).

Lemma 6. *If $p < p_0$, then $E_x {\tau_\beta}^p < \infty$ for every $x \in \mathcal{P}_\beta$.*

Proof. Let $1/\lambda_1$ be the first eigenvalue of the Green operator $G_B f(\tilde{x}) = \mathcal{E}_{\tilde{x}} \int_0^{\eta_B} f(Y_t)dt$, $\tilde{x} \in \mathbb{R}^{d-1}$, for our isotropic stable process Y_t in $\mathbb{R}^{d-1}$. Here, B is the unit ball in $\mathbb{R}^{d-1}$, η_D is the first exit time of Y from $D \subset \mathbb{R}^{d-1}$, and $\mathcal{P}$, $\mathcal{E}$, are, respectively, the distribution and expectation corresponding to Y. For $r > 0$, by *scaling*, $1/(r^{-\alpha}\lambda_1)$ is the eigenvalue for G_{rB} and $\mathcal{P}_{\tilde{x}}\{\eta_{rB} > t\} \le ce^{-tr^{-\alpha}\lambda_1}$, $0 < t < \infty$. Let $\mathcal{C}$ be the cylinder as in (3). For $t > 0$, fixed $x = (x_1, \tilde{x}) \in \mathcal{P}_\beta$ and $s \ge 1 \vee 2x_1$, we have by Lemma 5 that

$$\begin{aligned} \mathbf{P}_x\{\tau_\beta > t\} &= \mathbf{P}_x\{\tau_\beta > t\,,\ \tau_\beta = \tau_s\} + \mathbf{P}_x\{\tau_\beta > t\,,\ \tau_\beta > \tau_s\} \\ &\le \mathbf{P}_x\{\tau_{\mathcal{C}} > t\} + \mathbf{P}_x\{X_{\tau_s} \in \mathcal{P}_\beta\} \le ce^{-ts^{-\alpha\beta}\lambda_1} + cs^{-\alpha\beta p_0}\,. \end{aligned}$$

Let us take $s = t^{(1-\epsilon)/(\alpha\beta)}$, where $0 < \epsilon \le 1/2$. We get

$$\mathbf{P}_x\{\tau_\beta > t\} \le ce^{-t^\epsilon\lambda_1} + ct^{-(1-\epsilon)p_0} \le ct^{-(1-\epsilon)p_0}$$

for large t. Thus, letting $p = (1-2\epsilon)p_0$ we get

$$\mathbf{E}_x {\tau_\beta}^p = \int_0^\infty pt^{p-1}\mathbf{P}_x\{\tau_\beta > t\}\,dt \le const. + const. \int_1^\infty t^{-\epsilon p_0 - 1}dt < \infty\,.$$

This finishes the proof. $\square$

Proof of Theorem 1. The result follows from Lemma 2 and Lemma 6. $\square$

We conclude with three remarks.

Because of scaling of $\{X_t\}$, Theorem 1 holds with the same p_0 for the more general parabola-shaped regions of the form

$$\{x = (x_1, \tilde{x})\ :\ x_1 > 0\,,\ \tilde{x} \in \mathbb{R}^{d-1}\,, |\tilde{x}| < a\, x_1^\beta\},$$

for any $0 < a < \infty$.

If $\beta \downarrow 0$ then $\mathcal{P}_\beta$ approaches an infinite cylinder, for which the exit time has exponential moments (compare the proof of Lemma 6).

The second endpoint $\beta \uparrow 1$ suggests studying the rate of convergence (to 1) of the critical exponent of integrability of the exit time from the right circular cone as the opening of the cone tends to 0. A partial result in this direction is given in [13]. [1] contains more information on our stable processes in cones and a hint how to approach the problem.

Acknowledgment. The second author is grateful for the hospitality of the Departments of Statistics and Mathematics at Purdue University, where the paper was prepared in part. We are grateful to the referee for pointing out several corrections.

References

[1] R. Bañuelos and K. Bogdan, *Symmetric stable processes in cones*, Potential Analysis, 21 (2004), 263-288.

[2] R. Bañuelos and T. Carroll, *Sharp Integrability for Brownian Motion in Parabola-shaped Regions* (2003), J. Funct. Anal. (to appear).

[3] R. Bañuelos, R. D. DeBlassie and R. Smits, *The first exit time of planar Brownian motion from the interior of a parabola*, Ann. Prob. 29 (2001), 882-901.

[4] R. Bañuelos and R. Smits, *Brownian motion in cones*, Prob. Th. Rel. Fields, 108 (1997), 299-319.

[5] M. van den Berg, *Subexponential behavior of the Dirichlet heat kernel*, J. Funct. Anal., 198, 28-42.

[6] R. M. Blumenthal, R. K. Getoor and D.B. Ray, *On the distribution of first hits for the symmetric stable process*, Trans. Amer. Math. Soc. 99 (1961), 540–554.

[7] K. Bogdan and T. Jakubowski, *Problème de Dirichlet pour les fonctions α-harmoniques sur les domaines coniques* (2004), preprint.

[8] K. Bogdan and T. Żak, *On Kelvin transformation* (2004), preprint.

[9] K. Burdzy and T. Kulczycki, *Stable processes have thorns,* Ann. Probab. 31 (2003) 170–194.

[10] D. L. Burkholder, *Exit times of Brownian motion, harmonic majorization, and Hardy spaces*, Advances in Math. 26(2) (1977), 182–205.

[11] R. D. DeBlassie and R. Smits. *Brownian motion in twisted domains,* Transactions of the American Mathematical Society (to appear).

[12] R. DeBlassie, *The first exit time of a two–dimensional symmetric stable process from a wedge*, Ann. Prob., 18 (1990), 1034–1070.

[13] T. Kulczycki, *Exit time and Green function of cone for symmetric stable processes*, Probab. Math. Statist. 19(2) (1999), 337–374.

[14] W. Li, *The first exit time of Brownian motion from unbounded convex domains,* Annals of Probability, **31**, 1078–1096.

[15] M. Lifshitz and Z. Shi, *The first exit time of Brownian motion from parabolic domain,* Bernoulli 8(6) (2002), 745–767.

[16] P. J. Méndez-Hernández, *Exit times from cones in $\mathbb{R}^n$ of symmetric stable processes* Illinois J. Math 46 (2002), 155-163.

[17] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, University Press Springer-Verlag, Cambridge, 1999.

Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395

Institute of Mathematics, Polish Academy of Sciences, and Institute of Mathematics, Wrocław University of Technology, 50-370 Wrocław, Poland

E-mail address: `banuelos@math.purdue.edu` `bogdan@im.pwr.wroc.pl`